Transmission Line Equations

the telegrapher's equations are given as:

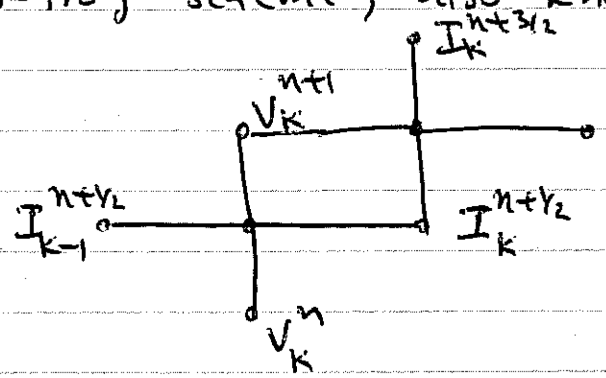
$$\text{PDEs}\begin{cases} \partial_x v(x,t) + L\,\partial_t i(x,t) + R\,i(x,t) = V_F(x,t) \\ \partial_x i(x,t) + C\,\partial_t v(x,t) + G\,v(x,t) = I_F(x,t) \end{cases}$$

$$0 < x < \ell$$
$$0 < t$$

$v(x,t)$ - voltage on the line
$i(x,t)$ - current on the line
L - p.u.l. inductance
C - p.u.l. capacitance
R - p.u.l. series resistance
G - p.u.l. shunt capacitance

$V_F(x,t)$ - distributed voltage source
$I_F(x,t)$ - distributed current source.

the computational molecule for the interlaced leap-frog scheme, also known as FDTD, is



$$\begin{cases} N\text{ - sections.} \\ \Delta x = \ell/N \end{cases}$$

$k = 1, \ldots, N+1$: $V_k^n \approx v((k-1)\Delta x, n\Delta t)$

$k = 1, \ldots, N$: $I_k^n \approx i((k-1/2)\Delta x, n\Delta t)$

Note: we're only using $1/2$-index notation for time.

Using 2^{nd}-Order centred finite differences for discretizing the PDE's :

$$\frac{V_{k+1}^{n+1} - V_k^{n+1}}{\Delta x} + L\,\frac{I_k^{n+3/2} - I_k^{n+1/2}}{\Delta t} + R\,\frac{I_k^{n+3/2} + I_k^{n+1/2}}{2} = \frac{V_{Fk}^{n+3/2} + V_{Fk}^{n+1/2}}{2}$$

$$\frac{I_{k+1}^{n+1/2} - I_{k-1}^{n+1/2}}{\Delta x} + C\,\frac{V_k^{n+1} - V_k^{n}}{\Delta t} + G\,\frac{V_k^{n+1} + V_k^{n}}{2} = \frac{I_{Fk}^{n+1} + I_{Fk}^{n}}{2}$$

Solving for $I_k^{n+3/2}$ and V_k^{n+1} :

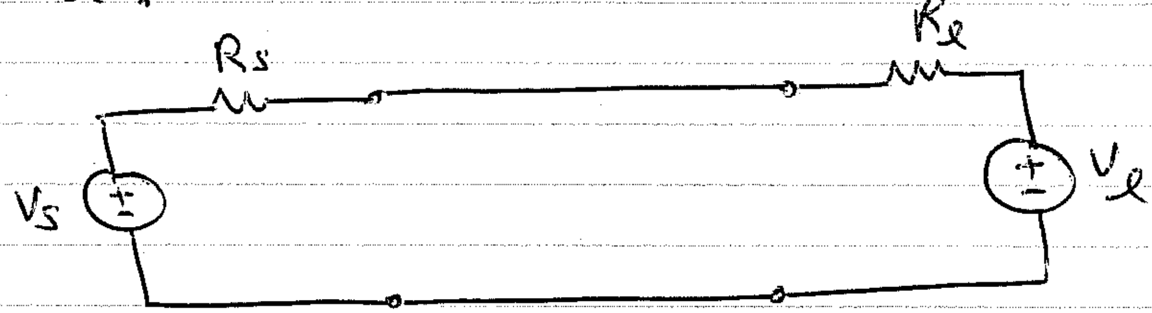
$$\left\{\begin{aligned}
\left(L\frac{\Delta x}{\Delta t} + \frac{R}{2}\Delta x\right) I_k^{n+3/2} &= \left(L\frac{\Delta x}{\Delta t} - \frac{R}{2}\Delta x\right) I_k^{n+1/2} - \left(V_{k+1}^{n+1} - V_k^{n+1}\right) \\
&\quad + \frac{\Delta x}{2}\left(V_{Fk}^{n+3/2} + V_{Fk}^{n+1/2}\right) \\
\left(C\frac{\Delta x}{\Delta t} + \frac{G}{2}\Delta x\right) V_k^{n+1} &= \left(C\frac{\Delta x}{\Delta t} - \frac{G}{2}\Delta x\right) V_k^{n} - \left(I_k^{n+1/2} - I_{k-1}^{n+1/2}\right) \\
&\quad + \frac{\Delta x}{2}\left(I_{Fk}^{n+1} + I_{Fk}^{n}\right)
\end{aligned}\right.$$

These are the general up-date equations.

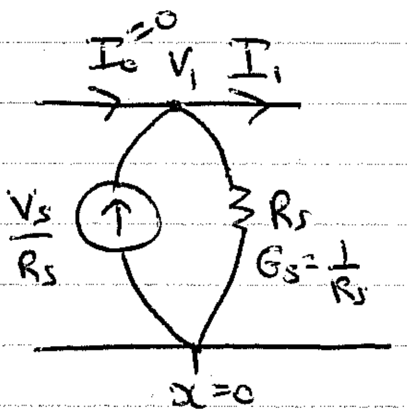
Now we need to look at incorporating the boundary conditions :

$$\left\{\begin{aligned}
v(0,t) &= V_s - R_s\, i(0,t) \\
v(\ell,t) &= V_\ell + R_\ell\, i(\ell,t)
\end{aligned}\right.$$

These boundary conditions represent Thévenin Equivalent sources at $x=0$ and $x=\ell$.



These can be converted to Norton equivalents in order to better incorporate them into the Finite-difference scheme:



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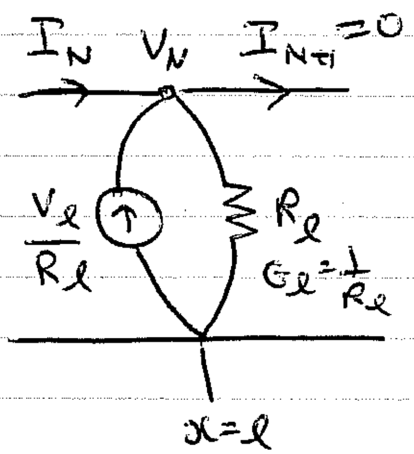

$I_0 = I_{N+1} = 0$ because the line ends at $x=0$, $x=\ell$, the first current is I_1 and the last current is I_N

Thus, at the first cell we have:

$$\begin{cases} I_0 = 0 \\ G_0^T = G_s/\Delta x = \dfrac{1}{R_s \Delta x} & \text{equivalent p.u.l conductance} \\ I_{F0}^T = \dfrac{V_s}{R_s \Delta x} & \text{equivalent p.u.l current source} \end{cases}$$

and at the final cell:

$$\begin{cases} I_N = 0 \\ G_N^T = \dfrac{G_\ell}{\Delta X} = \dfrac{1}{R_\ell \Delta X} & \text{equivalent p.u.l conductance} \\ I_{FN}^T = \dfrac{V_\ell}{R_\ell \Delta X} & \text{equivalent p.u.l current source} \end{cases}$$

G_0^T and G_N^T add to whatever conductance is already at these cells.

I_{F0}^T and I_{FN}^T also add to whatever current sources are already at these cells.

So the same up-date equation can be used after these replacements have been made.